A RIMS - Kyoto University &

"Arithmetic and Homotopic Galois Theory" Lecture

LECTURES ON THE GROTHENDIECK SECTION CONJECTURE

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The section conjecture was formulated by Grothendieck in 1983 in a letter to Faltings. For a field K and and a K-variety X the étale fundamental group $\pi_1^{et}(X)$ naturally surjects onto the absolute Galois group G_K of K; write

$$Sec(\pi_1^{et}(X) \twoheadrightarrow G_K)$$

for the set of conjugacy classes of sections of this surjection. By functoriality there is a natural map

$$X(K) \to Sec(\pi_1^{et}(X) \twoheadrightarrow G_K), \tag{1}$$

called the Kummer map. Grothendieck observed that when K is finitely generated over $\mathbb{Q}$ and X is a hyperbolic curve then the map (1) is injective. Further he speculated about the image of the map (1); in particular for X proper smooth curve of genus at least two he conjectured that the map (1) is in fact a bijection.

This statement has become known as *the Grothendieck section conjecture* and it remains one of the most important open problems in anabelian geometry. The section conjecture describes the set of K-rational points on X, an object of fundamental independent interest in arithmetic geometry, in terms of certain homomorphisms between profinite groups having close relationship to both arithmetic and topology.

In this series of three lectures we will discuss some aspects of the conjecture; the first talk is a general introduction, the second one explains relations between the section conjecture and the theory of obstructions of adelic points, finally the third talk various local-global principles for sections and is based on lecturer's own research.

We further provide a list of references of general interest for each of the lecture below.

Keywords: anabelian geometry, section conjecture, local-global obstructions

REFERENCES

[Gro97] A. Grothendieck. "Brief an G. Faltings". In: *Geometric Galois actions, 1.* Vol. 242. London Math. Soc. Lecture Note Ser. With an English translation on pp. 285–293. Cambridge Univ. Press, Cambridge, 1997, pp. 49–58.

[Sti13] J. Stix. *Rational points and arithmetic of fundamental groups. Evidence for the section conjecture.* English. Vol. 2054. Lect. Notes Math. Berlin: Springer, 2013.

OVERVIEW ON GALOIS SECTIONS — TALK 1

In this talk we give a general introduction to the topic of Galois sections for curves X over K. We will introuduce the Kummer map and discuss its injectivity for general fields K. Further we will construct sections from K-rational points lying on the boundary of X; these are called cuspidal sections.

Moreover we will discuss other variants (local and birational) of the section conjecture and show how they are related to each other.

Keywords: Kummer map, cuspidal section, birational section conjecture

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Selmer sections and obstructions — Talk 2

For a number field K and a K-variety there is a natural injection

$$X(K) \hookrightarrow X(\mathbb{A}_K)$$

of the set of K-rational points into the set of adelic points of X. The classical Hasse principle asks whether the existence of an adelic point already implies the existence of a K-rational point. There are more refined methods of finding obstructions (Brauer-Manin, descent, étale Brauer-Manin ...) which cut off closed subsets of the set of rational points; one can then ask a similar question whether their nonemptyness implies the existence of a K-rational on X.

In the second talk we will introduce other classes of sections called Selmer and locally geometric sections and discuss their close relationship to the theory of obstructions.

Keywords: Selmer sections, local-global obstructions, Brauer-Manin type obstructions

References

[BS25] L. A. Betts and J. Stix. "Galois sections and p-adic period mappings". In: *Ann. of Math. (2)* 201.1 (2025), pp. 79–166. URL: https://doi.org/10.4007/annals.2025.201.1.2.

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[LV20] B. Lawrence and A. Venkatesh. "Diophantine problems and p-adic period mappings". In: *Invent. Math.* 221.3 (2020), pp. 893–999. URL: https://doi-org.kyoto-u.idm.oclc.org/10.1007/s00222-020-00966-7.

[Sti11] J. Stix. "The Brauer-Manin obstruction for sections of the fundamental group". In: *J. Pure Appl. Algebra* 215.6 (2011), pp. 1371–1397. URL: https://doi-org.kyoto-u.idm.oclc.org/10.1016/j.jpaa.2010.08.017.

Local-global principle and reductions of sections — Talk 3

In the final talk we will state a local-global principle for sections, namely that the conjugacy class of a global section is uniquely determined by conjugacy classes of its localizations, and give a sketch of its proof. Then we will introduce two fundamental problems for Selmer sections: adelicity and local cuspidality, and how they are related to the global sections conjecture.

In the last part we will discuss the lecturer's work towards some partial results for the above two problems.

Keywords: local-global principle, reductions of sections, resolution of nonsingularities

References

[MT] S. Mochizuki and S. Tsujimura. "Resolution of Nonsingularities, Point-theoreticity, and Metric-admissibility for p-adic Hyperbolic Curves". preprint available on http://hdl.handle.net/2433/284398.

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[Por24] W. Porowski. "Locally conjugate Galois sections". In: *J. Reine Angew. Math.* 815 (2024), pp. 41–70. URL: https://doi.org/10.1515/crelle-2024-0047.

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[Tam04] A. Tamagawa. "Resolution of nonsingularities of families of curves". In: *Publ. Res. Inst. Math. Sci.* 40.4 (2004), pp. 1291–1336. URL: http://projecteuclid.org.kyoto-u.idm.oclc.org/euclid.prims/1145475448.

Schedule Location: RIMS Kyoto University, room 110.

- Talk 1: 22 April (Wed.) 10:00-12:30
- Talk 2: 23 April (Thu.) 10:00-12:30
- Talk 3: 24 April (Fri.) 13:30-16:00

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